Ecuación de Laplace

con condiciones de contorno

Problema de Dirichlet:

$$\begin{cases} \Delta u(\bar{x}) = 0 & \bar{x} \in \Omega \\ u(\bar{x}) = f(\bar{x}) & \bar{x} \in \partial\Omega \end{cases}$$

Si Ω es acotado abierto "decente" y f continua sobre $\partial\Omega \Rightarrow$ existe una única solución u, C^2 en Ω y continua en $\bar{\Omega} = \Omega \cup \partial\Omega$

Problema de Neumann:

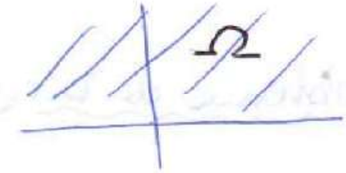
$$\begin{cases} \Delta u(\bar{x}) = 0 & \bar{x} \in \Omega \\ \dfrac{\partial u}{\partial n}(\bar{x}) = f(\bar{x}) & x \in \partial\Omega \end{cases}$$

Problema de Robin:

$$\begin{cases} \Delta u(\bar{x}) = 0 & \bar{x} \in \Omega \\ A\,u(\bar{x}) + B\dfrac{\partial u}{\partial n}(\bar{x}) = f(\bar{x}) & \bar{x} \in \partial\Omega \end{cases}$$

Sobre unicidad del problema de Dirichlet.

Si Ω es no acotado, la sol. no es única.



$h_1(x,y) = e^x \cos y$

$h_2(x,y) = e^x \cos y + \operatorname{sen} y \cosh x$

$h_3(x,y) = e^x \cos y + xy + e^x \operatorname{sen} y + y$

Son soluciones de: $\begin{cases} \Delta h = 0 \\ h(x,0) = e^x \end{cases}$

Sobre existencia de solución en Problema de Neuman:

si Ω es acotado, "decente".

Teo. Gauss:

$$0 = \iint_\Omega \Delta u \, dA = \iint_\Omega \nabla \cdot \nabla u \, dA = \int_{\partial\Omega} \nabla u \cdot n \, ds$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\partial\Omega} f(\bar{x}) \, ds$$

$\Rightarrow$ si se impone $\frac{\partial u}{\partial n} = f(\bar{x})$ en $\partial\Omega$, para que exista solución debe ser: $\int_{\partial\Omega} f(\bar{x}) ds = 0$

Armónicas en $\mathbb{R}^2$

Si h es armónica en $\Omega \subset \mathbb{R}^2$ y depende de una sola variable, tiene alguna de estas formas:

Cartesiana: $h(x,y) = Ax + B$

$h(x,y) = Ay + B$

Polares: $h(r,\theta) = A\theta + B$

$h(r,\theta) = A \ln r + B \quad (\text{si } 0 \notin \Omega)$

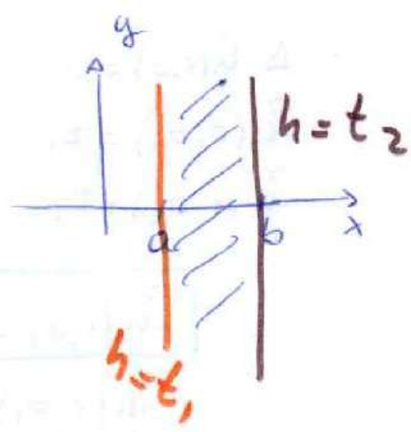
Problemas de Dirichlet nivel principiante

(A) $D = \{(x,y) : a < x < b\}$

$\Delta h(x,y) = 0$

$h(a,y) = t_1$

$h(b,y) = t_2$

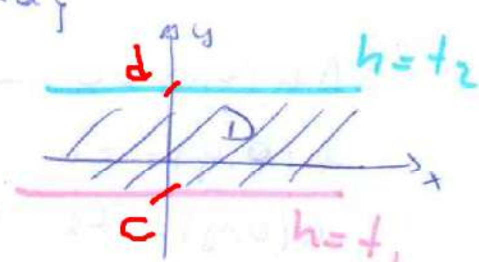


$$\boxed{h(x,y) = Ax + B}$$

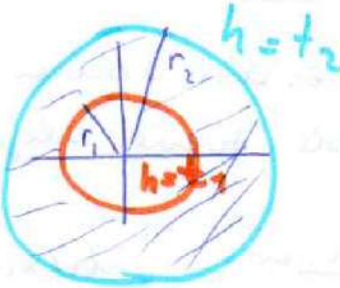
$$\left.\begin{aligned} h(a,y) &= A.a + B = t_1 \\ h(b,y) &= Ab + B = t_2 \end{aligned}\right\} \rightarrow A, B$$

(B) Similarmente si $D = \{(x,y) : c < y < d\}$

$$\boxed{h(x,y) = Ay + B}$$



(c) $D = \{(x,y) : r_1 < \sqrt{x^2+y^2} < r_2\}$ $\quad r_1 > 0$



$\Delta \tilde{h}(r,\theta) = 0$ en D

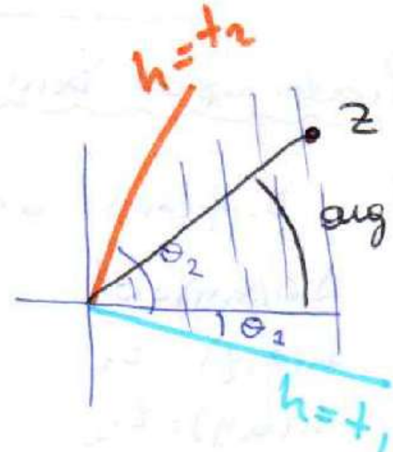
$\tilde{h}(r_1,\theta) = t_1$

$\tilde{h}(r_2,\theta) = t_2$

$$\boxed{\tilde{h}(r,\theta) = A \ln r + B}$$

$$\left.\begin{array}{l} h(r_1,\theta) = A \ln r_1 + B = t_1 \\ h(r_2,\theta) = A \ln r_2 + B = t_2 \end{array}\right\} \longrightarrow A, B.$$

(D) $D = \{(x,y) : e1 < \arg((x+iy) < e2 \}$



$\Delta \tilde{h}(r,\theta) = 0$ en D

$\tilde{h}(r,\theta_1) = t_1$

$\tilde{h}(r,\theta_2) = t_2$

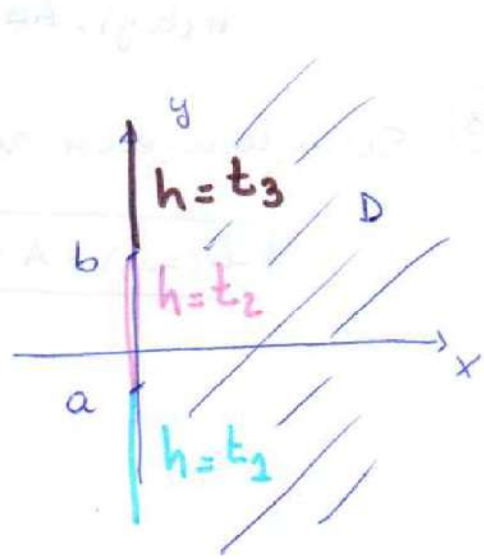
$$\boxed{\tilde{h}(r,\theta) = A\theta + B}$$

$$\left.\begin{array}{l} \tilde{h}(r,\theta_1) = A\theta_1 + B = t_1 \\ \tilde{h}(r,\theta_2) = A\theta_2 + B = t_2 \end{array}\right\} \longrightarrow A, B$$

Si $-\pi/2 < \theta < \pi/2$:

$\theta = \operatorname{arctg}\left(\frac{y}{x}\right)$

(D') $D = \{(x,y) : x > 0\}$

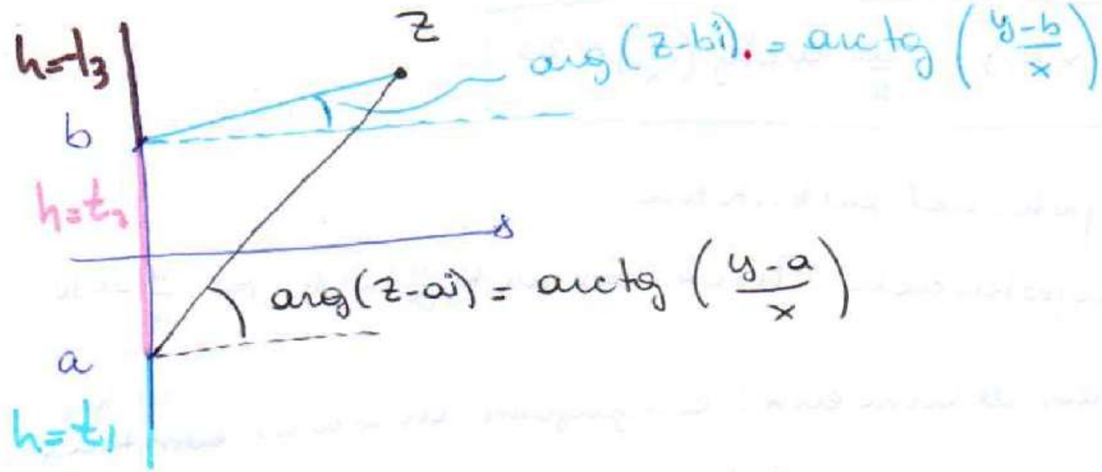


$\Delta h(x,y) = 0$ en D

$h(0,y) = t_1 \quad y < a$

$h(0,y) = t_2 \quad a < y < b$

$h(0,y) = t_3 \quad b < y$



Ptos con $x=0,\ y<a$: $\arg(z-ai) = -\pi/2$
$\arg(z-bi) = -\pi/2$

$x=0 \quad a<y<b$: $\arg(z-ai) = \pi/2$
$\arg(z-bi) = -\pi/2$

$x=0 \quad y>b$ $\quad \arg(z-ai) = \pi/2$
$\arg(z-bi) = \pi/2$

Solución:

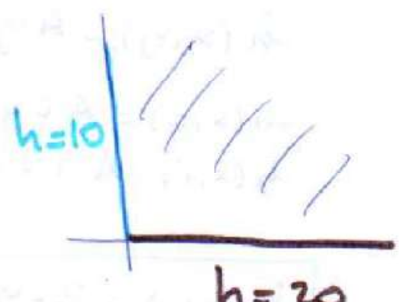
$$h(r,\theta) = A \arg(z-ai) + B \arg(z-bi) + C$$

$$\left.\begin{aligned} &\rightarrow t_1 = A(-\pi/2) + B(-\pi/2) + C \\ &\rightarrow t_2 = A(\pi/2) + B(-\pi/2) + C \\ &\rightarrow t_3 = A(\pi/2) + B(\pi/2) + C \end{aligned}\right\} \rightarrow A, B, C.$$

$$\boxed{h(x,y) = A \cdot \operatorname{arctg}\left(\frac{y-a}{x}\right) + B \operatorname{arctg}\left(\frac{y-b}{x}\right) + C}$$

Ejemplo

$$\begin{cases} \Delta h = 0 & x>0, y>0 \\ h(0,y) = 10 & y>0 \\ h(x,0) = 20 & x>0 \end{cases}$$



$\tilde{h}(r,\theta) = A\theta + B \qquad \tilde{h}(r,0) = A\cdot 0 + B = 20$
$\tilde{h}(r,\pi/2) = A\pi/2 + B = 10 \quad \rightarrow A = -\frac{20}{\pi} \quad B = 20$

$$\Rightarrow \boxed{h(x,y) = -\frac{20}{\pi}\,\operatorname{arctg}\left(\frac{y}{x}\right) + 20}$$

Si h es potencial electrostático:

equipotenciales: $h = cte \iff \operatorname{arctg}\left(\frac{y}{x}\right) = cte \iff \underbrace{\frac{y}{x} = cte}_{\text{rectas}}$

líneas de corriente: conjuntos de nivel de conjugado armónico

conj. armónico:

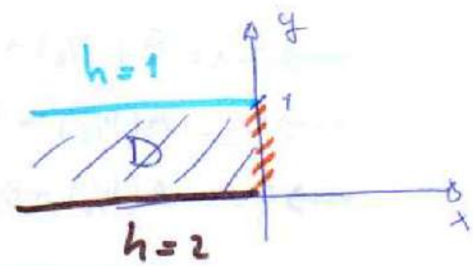
$$\varphi(x,y) = +\frac{20}{\pi}\ln\left(\sqrt{x^2+y^2}\right)$$

líneas de corriente: $\sqrt{x^2+y^2} = cte \rightarrow$ circunferencias

$\rightarrow$ líneas de corriente.

Ejemplo

$$\begin{cases} \Delta h = 0 & \text{en } D \\ h(x,0) = 2 & x<0 \\ h(x,1) = 1 & x<0 \\ \frac{\partial h}{\partial n}(0,y) = 0 & 0<y<1 \end{cases}$$



$$h(x,y) = Ay + B \quad \Rightarrow \quad \frac{\partial h}{\partial n}(0,y) = \frac{\partial h}{\partial x}(0,y) = 0$$

$$\left.\begin{array}{l} h(x,0) = A\cdot 0 + B = 2 \\ h(x,1) = A\cdot 1 + B = 1 \end{array}\right\} \Rightarrow B = 2,\ A = -1$$

$$\boxed{h(x,y) = 2 - y}$$

Ejemplo

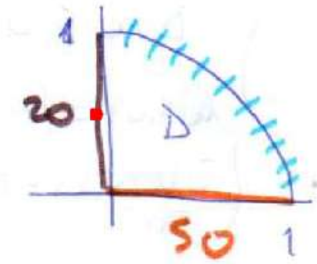
Resolver:

$\Delta h(x,y) = 0 \qquad (x,y) \in D$

$h(x,0) = 50 \qquad 0<x<1$

$h(0,y) = 20 \qquad 0<y<1$

$\dfrac{\partial h}{\partial n}(x,y) = 0 \qquad$ si $x^2+y^2=1,\ x>0, y>0$



se puede transforma en el problema anterior con la transformación
$w = \frac{2}{\pi}\log(z) = \frac{2}{\pi}\ln|z| + i\frac{2}{\pi}\arg(z)$

$\dfrac{\partial h}{\partial n} = \dfrac{\partial h}{\partial r}$

en circunferencia

$\tilde{h}(r,\theta) = A\theta + B \quad \to$ armónica. Debe ser $\tilde{h} = 50$ si $\theta = 0$, $\tilde{h} = 20$ si $\theta = \pi/2$

$\tilde{h}(r,0) = A\cdot 0 + B = 50$

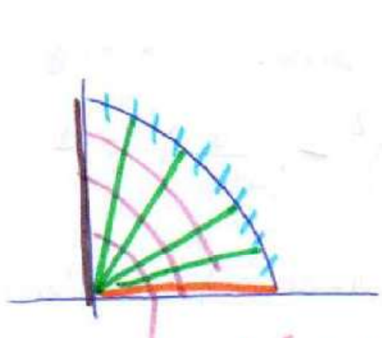
$\tilde{h}(r,\pi/2) = A\frac{\pi}{2} + B = 20$

$\Rightarrow B = 50, \quad A = -\frac{60}{\pi}$

$\tilde{h}(r,\theta) = -\frac{60}{\pi}\theta + 50$

$$\boxed{h(x,y) = -\frac{60}{\pi}\arctan\left(\frac{y}{x}\right) + 50}$$

Isoterma: $h(x,y) = cte \iff \frac{y}{x} = cte \iff y = cte\cdot x$



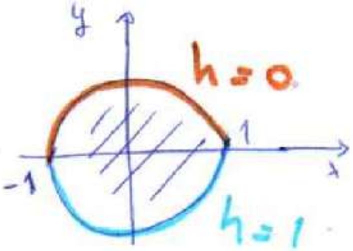
líneas de flujo

y si no son regiones tan fáciles ...?

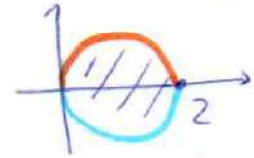
transformaciones conformes!

Ejemplo

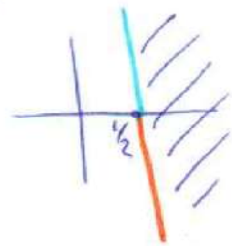
$$\begin{cases} \Delta h(x,y) = 0 & x^2+y^2<1 \\ h(x,y)=0 & \text{si } x^2+y^2=1,\ y>0 \\ h(x,y)=1 & \text{si } x^2+y^2=1,\ y<0 \end{cases}$$



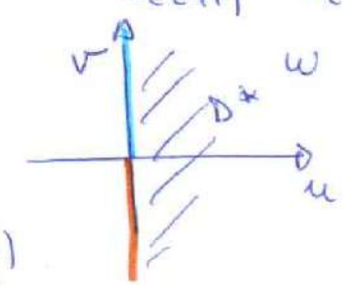
Transformamos: $z_1 = z+1$



$$z_2 = \frac{1}{z_1} = \frac{1}{z+1}$$



$$w = z_3 = z_2 - 1/2 = \frac{1}{z+1} - 1/2 = \frac{2-(z+1)}{2(z+1)} = \frac{1-z}{2(z+1)}$$



$$w = \frac{1-z}{2(z+1)} = \frac{(1-z)(\bar z+1)}{2\,|z+1|^2} =$$

$$= \frac{1}{2}\left(\frac{\bar z + 1 - z\bar z - z}{|z+1|^2}\right) = \frac{1}{2}\left(\frac{1-2yi-x^2-y^2}{(x+1)^2+y^2}\right)$$

$z - \bar z = 2yi$

$$\boxed{w = \underbrace{\frac{1-x^2-y^2}{2(x+1)^2+2y^2}}_{u(x,y)} - \underbrace{\frac{2yi}{2(x+1)^2+2y^2}}_{v(x,y)}}$$

Sol. en D^*: $\tilde H(r,\theta) = A\theta + B$ con $\left.\begin{array}{l} A\cdot\left(-\frac{\pi}{2}\right)+B = 0 \\ A\left(\pi/2\right)+B = 1 \end{array}\right\} \rightarrow \begin{array}{l} B = 1/2 \\ A = \frac{1}{\pi} \end{array}$

$\theta = \arg w = \operatorname{arctg}\left(\frac{v}{u}\right)$

$$H(u,v) = \frac{1}{\pi}\operatorname{arctg}\left(\frac{v}{u}\right) + \frac{1}{2}$$

$$\boxed{h(x,y) = \frac{1}{\pi}\operatorname{arctg}\left(\frac{-y.2}{1-x^2-y^2}\right) + \frac{1}{2}}$$

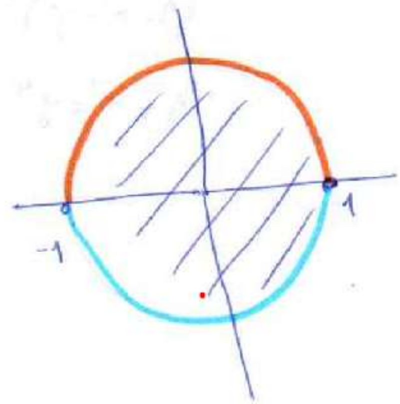
Veamos:

si $(x,y) \to$ circunf, $y>0$

$$h(x,y)=\frac{1}{\pi}\,\arctan\left(\underbrace{\frac{\overbrace{-y}^{<0}.2}{\underbrace{1-x^2-y^2}_{\to 0^+}}}_{\to -\infty}\right)+\frac{1}{2}$$

$\to -\pi/2$

$$\longrightarrow \frac{1}{\pi}\left(-\frac{\pi}{2}\right)+\frac{1}{2} = 0$$



si $(x,y) \longrightarrow$ circunf. $y<0$

$$h(x,y)=\frac{1}{\pi}\,\arctan\left(\underbrace{\frac{\overbrace{-y}^{>0}.2}{\underbrace{1-x^2-y^2}_{\to 0^+}}}_{\to +\infty}\right)+\frac{1}{2}$$

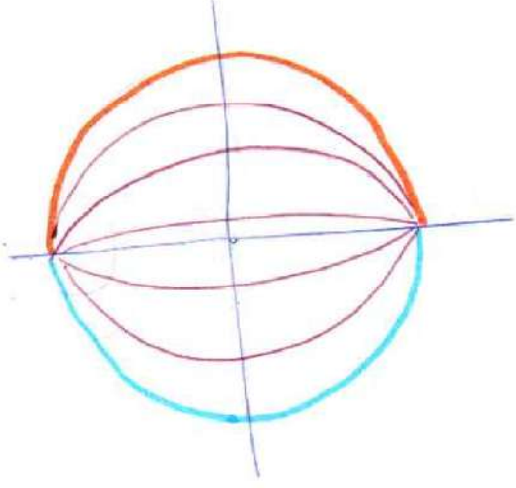
$\to \pi/2$

$$\frac{1}{\pi}\left(\frac{\pi}{2}\right)+\frac{1}{2} = 1$$

Si h es temperatura: isotermas: $h(x,y)=cte$

$$\iff \frac{-y.2}{1-x^2-y^2}=k \underset{\text{si } k\neq 0}{\Longrightarrow} \frac{-y.2}{k} = 1-x^2-y^2 \Rightarrow$$

$$\boxed{x^2+\left(y-\frac{1}{k}\right)^2 = 1+\frac{1}{k^2}}$$



Líneas de flujo: conjuntos de nivel de conj. armónica.

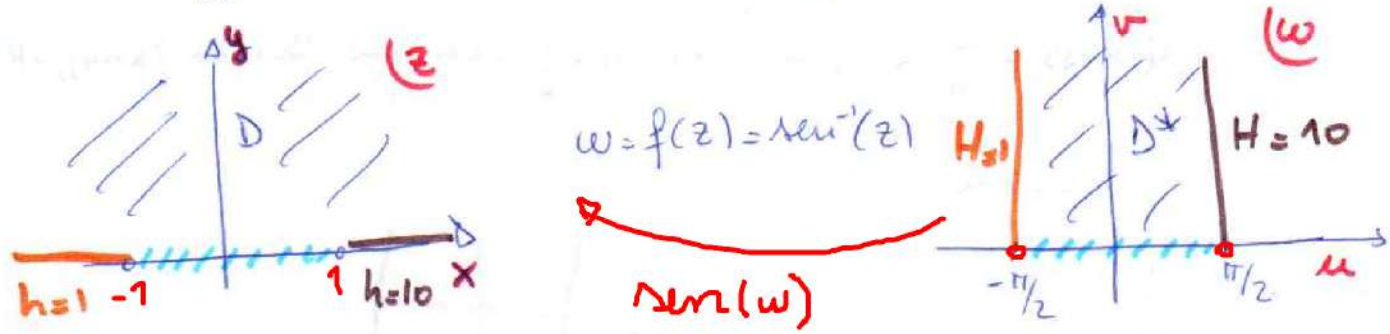
$$\varphi(x,y)=\frac{1}{\pi}\ln\left(\sqrt{u(x,y)^2+v^2(x,y)}\right)+\frac{1}{2}$$

$$\varphi(x,y)=cte \iff u^2(x,y)+v^2(x,y)=cte$$

Resolver

Distribución temperatura estado estacionario en región $D=\{(x,y)\in\mathbb{R}^2 : x>0\}$ si frontera $x<-1, y=0$ se mantiene a temp $=1$ y $(x>1, y=0)$ se mantiene a temp$=10$ y frontera $(y=0, |x|<1\}$ está aislada.

$$
\begin{array}{ll}
\Delta h(x,y)=0 & x>0, -\infty<y<\infty \\
h(x,0)=1 & x<-1 \\
h(x,0)=10 & x>1 \\
\dfrac{\partial h}{\partial n}(x,0)=0 & |x|<1
\end{array}
$$



Buscamos $H(u,v)$ armónica en D^* tal que:

$$
\left.\begin{array}{ll}
H(-\pi/2, v)=1 & v>0 \\
H(\pi/2, v)=10 & v>0 \\
\dfrac{\partial H}{\partial n}(u,0)=\dfrac{\partial H}{\partial v}(u,0)=0 &
\end{array}\right\} \Rightarrow H(u,v)=Au+B
$$

$$H\left(-\tfrac{\pi}{2},v\right)=A\left(-\tfrac{\pi}{2}\right)+B=1$$

$$H\left(\tfrac{\pi}{2},v\right)=A\,\tfrac{\pi}{2}+B=10$$

$$\Rightarrow A=9/\pi \qquad B=\frac{11}{2}$$

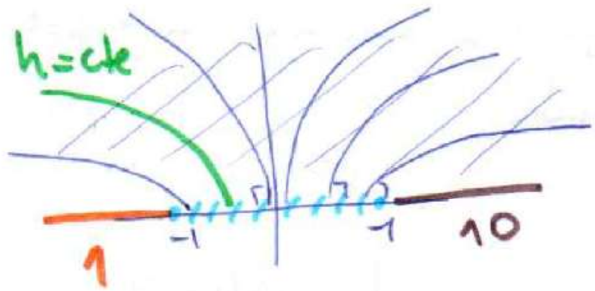
$$H(u,v)=\frac{9}{\pi}u+\frac{11}{2}$$

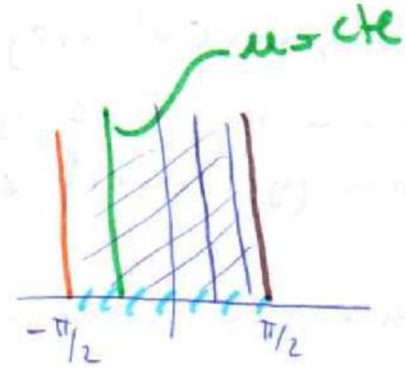
Pero: $w=u+iv=\text{sen}^{-1}(x+iy) \Rightarrow$

$$\boxed{h(x,y)=\frac{9}{\pi}\,\text{Re}\left(\text{sen}^{-1}(x+iy)\right)+\frac{11}{2}}$$

Isotermas? $h(x,y) = cte \iff Re(\operatorname{sen}^{-1}(x+iy)) = cte$.

$u = cte$.

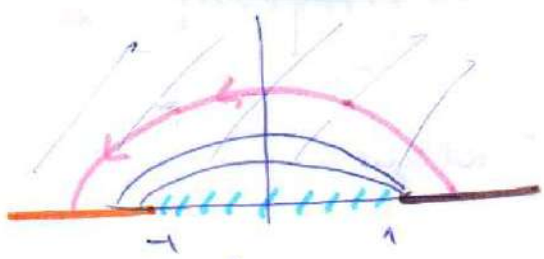


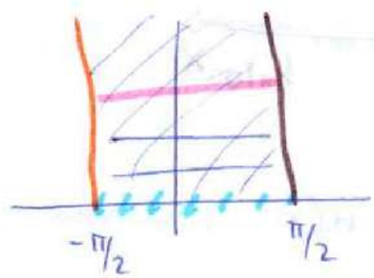


líneas de corriente / líneas de flujo de calor? curvas de nivel de la conj. armónica de h:

$$\varphi(x,y) = \frac{9}{\pi}\, Im\left(\operatorname{sen}^{-1}(x+iy)\right) + \frac{11}{2} = cte \iff Im\left(\operatorname{sen}^{-1}(x+iy)\right) = cte$$

$v = cte$.





línea de flujo paralela a frontera aislada.